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**by John D. Clayton**

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# AN INTERNAL VARIABLE THEORY OF DEFORMATION, DAMAGE, AND FRAGMENTATION OF SOLIDS

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**ABSTRACT:** A unified framework of continuum elasticity, inelasticity, damage mechanics, and fragmentation in solid materials is presented. A free energy functional accounts for thermodynamics of elastic deformation and damage, with kinetic relations for inelastic rates adapted from prior literature. Average fragment velocities follow from momentum conservation, average fragment sizes are deduced from an energy balance, and size and velocity distributions are constructed following entropy maximization. The model is further developed to study concrete subjected to ballistic loading scenarios.

**INTRODUCTION:** Constitutive models of dynamic deformation, damage evolution, and fragmentation of solid materials are needed in order to describe complex physical phenomena occurring, for example, in solid body collisions and ballistic impacts. Previous research efforts have often relied on a patchwork of separate models consisting of an equation-of-state for pressure, plasticity model for deviatoric strength, damage model for stress reduction/elimination, and fragmentation model for average fragment size [Johnson *et al.*, 1997; Raftenberg, 1998; Silling, 2001]. Such patchwork implementations may lack consistency from a kinematic and thermodynamic point-of-view. The aim of the present study is development of a self-consistent theory accounting for the above mechanisms, specifically amenable to brittle, crushable solids such as concrete, mortar, and cinder block. Static and dynamic properties of concrete follow from previous investigations [Holmquist *et al.*, 1993], while fragment sizes and distributions are predicted via an approach accounting for momentum and energy conservation [Grady, 1982], history-dependence of damage accumulation [Miller *et al.*, 1999], and maximum disorder principles [Englman *et al.*, 1987; Grady & Winfree, 2001].

**PROCEDURES, RESULTS AND DISCUSSION:** Fundamental aspects of the model framework for a deforming and fragmenting solid are presented here. The kinematic description begins with a multiplicative decomposition of the deformation gradient:

$$F_{.A}^a = \partial x^a / \partial X^A = F_{.a}^{Ea} F_{.A}^{D\alpha}, \quad (1)$$

where  $x^a$  and  $X^A$  denote spatial and reference coordinates,  $F_{.a}^{Ea}$  is the recoverable elastic deformation, and  $F_{.A}^{D\alpha}$  is the irreversible deformation associated with defects such as micro-cracks, voids, dislocations, or shear discontinuities evolving within the material. The spatial velocity gradient then follows directly from (1) as

$$L_{.b}^a = \partial \dot{x}^a / \partial x^b = \dot{F}_{.a}^{Ea} F^{E-1a}_{.b} + F_{.a}^{Ea} \dot{F}_{.B}^{D\alpha} F^{D-1B}_{.b} F^{E-1\beta}_{.b}, \quad (2)$$

with the superposed "□" a material time derivative. Standard balances of linear and angular momentum apply. The Helmholtz free energy, on a per unit intermediate configuration volume basis, is assumed to exhibit the following general dependencies:

$$\tilde{\psi} = \tilde{\psi}(E_{\alpha\beta}^E, \theta, \xi^{(k)}), \quad (3)$$

where the elastic strain  $2E_{\alpha\beta}^E = F_{.a}^{Ea} g_{ab} F_{.b}^{Eb} - \delta_{\alpha\beta}$ , with  $g_{ab}$  the spatial metric,  $\theta$  the absolute temperature, and  $\xi^{(k)}$  a dimensionless set of internal variables. Stress-strain, temperature-entropy, and internal dissipation relations are then deduced as

$$\sigma^{ab} = J^{E-1} F_{.a}^{Ea} \frac{\partial \tilde{\psi}}{\partial E_{\alpha\beta}^E} F_{.b}^{Eb}, \quad \tilde{\eta} = -\partial_{\theta} \tilde{\psi}, \quad P_{.b}^{\alpha} L_{.a}^{D\beta} - \partial_{\xi^{(k)}} \tilde{\psi} \dot{\xi}^{(k)} \geq 0, \quad (4)$$

with  $P_{.b}^{\alpha} = J^E F_{.a}^{E-1a} (\sigma_{.b}^a - \psi \delta_{.b}^a) F_{.b}^{Eb}$ . Here,  $\sigma^{ab}$  is the symmetric Cauchy stress,  $J^E$  is the elastic Jacobian,  $\psi$  is the free energy per unit current volume,  $\tilde{\eta}$  is the entropy per unit intermediate volume, and  $L_{.a}^{D\beta} = \dot{F}_{.B}^{D\beta} F^{D-1B}_{.a}$  is the purely inelastic velocity gradient.

The framework is hereafter specialized to brittle crushable solids, with  $F_{.A}^{D\alpha}$  now associated with crack opening and sliding, as well as pore collapse during compression. Let  $\tilde{A}$  represent the cumulative local micro-cracked area per unit intermediate volume  $\tilde{V}$ , such that  $D = \xi^{(1)} \propto \tilde{A} \tilde{V}^{1/3}$ , and restrict  $0 \leq D \leq 1$ . Furthermore, let  $\varphi = \xi^{(2)} = J^{D-1} - 1$  represent volume reduction upon crushing. The specific free energy density then is postulated as

$$\tilde{\psi} = \frac{1}{2} K_1 (E_{.a}^{Ea})^2 + \frac{1}{3} K_2 (E_{.b}^{Eb})^3 + \frac{1}{4} K_3 (E_{.c}^{Ec})^4 + G(1-D) \hat{E}_{\delta\epsilon}^E \hat{E}^{E\delta\epsilon} + \Gamma(D) + Y(\theta), \quad (5)$$

where  $K_1$ ,  $K_2$ , and  $K_3$  depend on  $\varphi$  and  $D$ ,  $G$  is the shear modulus,  $\hat{E}_{\alpha\beta}^E$  is the elastic strain deviator,  $\Gamma$  accounts for surface and concentrated energy of micro-cracks, and  $Y$  describes the specific heat content. Deviatoric plasticity follows from the potential  $\Phi$ :

$$\dot{\hat{\epsilon}}_{ab}^D = \lambda \frac{\partial \Phi}{\partial \hat{\sigma}^{ab}}, \quad \Phi = \left[ A(1-D) + B(p/p_0)^N \right] \left[ 1 + C \ln(\lambda/\dot{\epsilon}_0) \right] \sigma_0, \quad (6)$$

where  $\dot{\hat{\epsilon}}_{ab}^D$  is the spatial deviatoric inelastic strain rate,  $3\lambda^2 = 2\dot{\hat{\epsilon}}_{ab}^D \dot{\hat{\epsilon}}^{Dba}$  for stresses exceeding the elastic limit,  $\hat{\sigma}^{ab}$  is the deviatoric stress,  $-3p = \sigma_{.a}^a$ , and  $A$ ,  $B$ ,  $C$ ,  $p_0$ ,  $N$ ,  $\dot{\epsilon}_0$ , and  $\sigma_0$  are material parameters [Holmquist *et al.*, 1993]. In the present

implementation, we assume small elastic shape changes and material isotropy such that the inelastic spin may be neglected. Porosity and damage evolution are controlled via

$$\varphi = \begin{cases} 0 & (p \leq p_c); (p \geq p_l), \\ \alpha p & (p_c < p < p_l), \end{cases} \quad \dot{D} = \hat{\kappa} \lambda \langle 1 - \pi_D (p/p_0) \rangle, \quad (7)$$

where  $\alpha$ ,  $\hat{\kappa}$ , and  $\pi_D$  are positive constants, and the bracket notation  $\langle x \rangle = x \forall x > 0$  and  $\langle x \rangle = 0 \forall x \leq 0$ . Fragmentation of the local volume element is assumed to occur when the damage reaches a critical magnitude,  $D = D_f$ . At this instant, a spatial energy balance relating intact and fragmented configurations is written, in rate form, as

$$d(U + E_c)/dt = d\left(\sum(u + e_l + e_s) + E_c\right)/dt, \quad (8)$$

where  $U = J^{E-1}(\tilde{\psi} + \tilde{\eta}\theta)$ ,  $u$  is the internal energy per fragment,  $e_l$  and  $e_s$  are energy due to the relative linear and spin momentum per fragment, and  $E_c$  is the equivalent kinetic energy of both the continuum element and the center-of-mass of the fragment cloud. Summation is implied over all fragments in (8), which yields the average fragment dimension  $b$ . Subscale distributions of cumulative fragment number per unit volume ( $N$ ) and velocity relative to the center-of-mass follow from statistical mechanics principles [Grady & Winfree, 2001], i.e.  $N = N_0 \exp(-N_0 b^3)$  where  $N_0 = N_f/V$  is the mean number of fragments per unit volume, while spatial-nonlocal averaging methods describe fragments larger than grid sizes in numerical implementations.

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